

# Lesson 4: Finding Arc Lengths and Areas of Sectors

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## Common Core Georgia Performance Standard

MCC9–12.G.C.5

### Essential Questions

1. How is radian measure related to degree measure?
2. How is an arc length related to the circumference of a circle?
3. How is the area of a sector related to the area of a circle?

### WORDS TO KNOW

<b>arc length</b>	the distance between the endpoints of an arc; written as $m\widehat{AB}$
<b>central angle</b>	an angle with its vertex at the center of a circle
<b>circumference</b>	the distance around a circle; $C = 2\pi r$ , where $C$ represents circumference and $r$ represents radius
<b>radian</b>	the measure of the central angle that intercepts an arc equal in length to the radius of the circle; $\pi$ radians = $180^\circ$
<b>radian measure</b>	the ratio of the arc intercepted by the central angle to the radius of the circle
<b>sector</b>	a portion of a circle bounded by two radii and their intercepted arc

## Recommended Resources

- IXL Learning. “Circles: Arc measure and arc length.”

<http://www.walch.com/rr/00057>

This interactive website gives a series of problems and scores them immediately. If the user submits a wrong answer, a description and process for arriving at the correct answer are provided. These problems start with a diagram of a circle with a given radius. Users are given an arc measure in degrees and are asked to find the length of the arc.

- IXL Learning. “Circles: Area of sectors.”

<http://www.walch.com/rr/00058>

This interactive website also gives a series of problems and scores them immediately, providing instant feedback if a wrong answer is submitted. These problems start with a diagram of a circle with a given radius. Users are given a sector with a central angle measure in degrees and are asked to find the area of the sector.

- Math Open Reference. “Arc Length.”

<http://www.walch.com/rr/00059>

This site gives a brief overview of how to calculate the arc length of a circle and provides a virtual manipulative to experiment with choosing different arc lengths. The equation is updated in real time, allowing users to see what changes in the equation and what remains the same.

- Problems with a Point. “Spinning wheel 1.”

<http://www.walch.com/rr/00060>

This sequence of problems introduces the concept of a radian through an application problem about a bicycle wheel. Users are asked a series of questions to explore the meaning of radian measure. Users are then offered hints to discover how to convert between radian measure and degree measure. Answers are provided at the bottom of the webpage.

- TeacherTube. “Geo Screencast: Arc Length.”

<http://www.walch.com/rr/00061>

This video explains how to find arc length given the radius of the circle and the degree measure of the central angle.

- TeacherTube. “Geo Screencast: Sector Area.”

<http://www.walch.com/rr/00062>

This video offers a tutorial on how to find the area of a sector given the radius of the circle and the degree measure of the central angle.

- Texas Instruments. “Radian Measure.”

<http://www.walch.com/rr/00063>

This lesson plan for an activity on the TI-Nspire guides users through a discovery lesson in which they learn how the intercepted arc is related to the central angle and the radius of the circle.

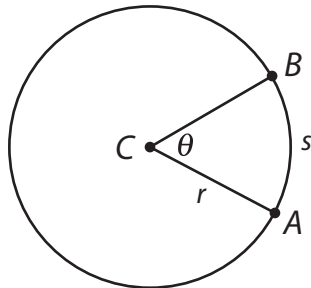
## Lesson 3.4.1: Defining Radians

### Introduction

All circles are similar; thus, so are the arcs intercepting congruent angles in circles. A central angle is an angle with its vertex at the center of a circle. We have measured an arc in terms of the central angle that it intercepts, but we can also measure the length of an arc. **Arc length**, the distance between the endpoints of an arc, is proportional to the radius of the circle according to the central angle that the arc intercepts. The constant of proportionality is the radian measure of the angle. You already know how to measure angles in degrees. Radian measure is another way to measure angles. An angle measure given in degrees includes a degree symbol. An angle measure given in radians does not.

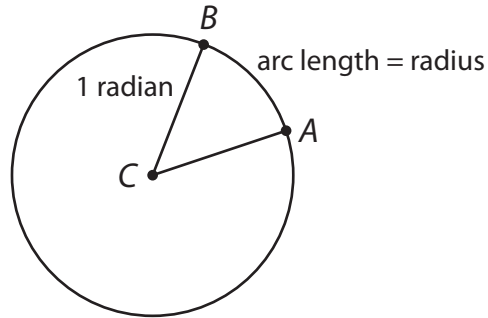
### Key Concepts

- Arc length is the distance between the endpoints of an arc, and is commonly written as  $m\widehat{AB}$ .
- The **radian measure** of a central angle is the ratio of the length of the arc intercepted by the angle to the radius of the circle.



- The definition of radian measure leads us to a formula for the radian measure of a central angle  $\theta$  in terms of the intercepted arc length,  $s$ , and the radius of the circle,  $r$ :  $\theta = \frac{s}{r}$ .

- When the intercepted arc is equal in length to the radius of the circle, the central angle measures **1 radian**.



- Recall that the circumference, or the distance around a circle, is given by  $C = 2\pi r$ , where  $C$  represents circumference and  $r$  represents radius.
- Since the ratio of the arc length of the entire circle to the radius of the circle is  $\frac{2\pi r}{r} = 2\pi$ , there are  $2\pi$  radians in a full circle.
- We know that a circle contains  $360^\circ$  or  $2\pi$  radians. We can convert between radian measure and degree measure by simplifying this ratio to get  $\pi$  radians =  $180^\circ$ .
- To convert between radian measure and degree measure, set up a proportion.

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180^\circ}$$

- To find the arc length when the central angle is given in radians, use the formula for radian measure to solve for  $s$ .
- To find arc length  $s$  when the central angle is given in degrees, we determine the fraction of the circle that we want to find using the measure of the angle. Set up a proportion with the circumference,  $C$ .

$$\frac{s}{C} = \frac{\text{degree measure}}{360^\circ}$$

## Guided Practice 3.4.1

### Example 1

Convert  $40^\circ$  to radians.

1. Set up a proportion.

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180^\circ}$$
$$\frac{x}{\pi} = \frac{40^\circ}{180^\circ}$$

2. Multiply both sides by  $\pi$  to solve for  $x$ .

$$x = \frac{40\pi}{180} = \frac{2\pi}{9} \text{ radians}$$



### Example 2

Convert  $\frac{3\pi}{4}$  radians to degrees.

1. Set up a proportion.

$$\frac{\text{radian measure}}{\pi} = \frac{\text{degree measure}}{180^\circ}$$
$$\frac{\frac{3\pi}{4}}{\pi} = \frac{x}{180^\circ}$$
$$\frac{3}{4} = \frac{x}{180^\circ}$$

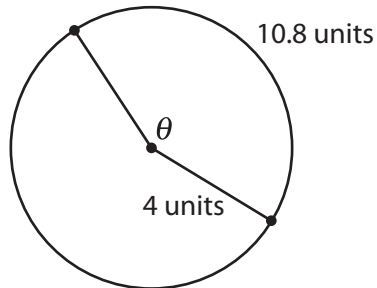
2. Multiply both sides by 180 to solve for  $x$ .

$$x = \frac{3(180)}{4} = 135^\circ$$



### Example 3

A circle has a radius of 4 units. Find the radian measure of a central angle that intercepts an arc of length 10.8 units.



1. Substitute the known values into the formula for radian measure.

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r} = \frac{10.8}{4}$$



2. Simplify.

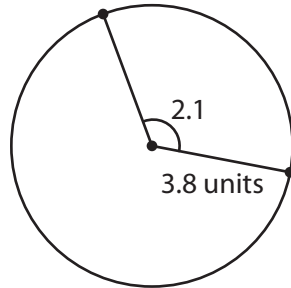
$$\theta = 2.7 \text{ radians}$$

The radian measure is 2.7 radians.



#### Example 4

A circle has a radius of 3.8 units. Find the length of an arc intercepted by a central angle measuring 2.1 radians.



1. Substitute the known values into the formula for radian measure.

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}$$

$$2.1 = \frac{s}{3.8}$$



2. Multiply both sides by 3.8 to solve for arc length.

$$s = 2.1 \cdot 3.8 = 7.98 \text{ units}$$

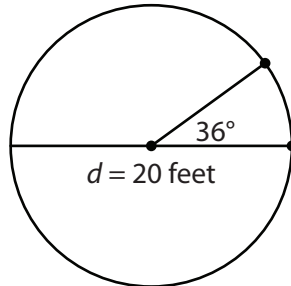
The arc length is 7.98 units.





### Example 5

A circle has a diameter of 20 feet. Find the length of an arc intercepted by a central angle measuring  $36^\circ$ .



1. Find the circumference of the circle.

$$\text{circumference} = \pi \cdot \text{diameter}$$

$$C = \pi d = 20\pi \text{ feet}$$

2. Set up a proportion.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{degree measure}}{360^\circ}$$

$$\frac{s}{C} = \frac{\text{degree measure}}{360^\circ}$$

$$\frac{s}{20\pi} = \frac{36}{360}$$

3. Multiply both sides by  $20\pi$  to find the arc length.

$$s = \frac{36}{360} \cdot 20\pi = \frac{1}{10} \cdot 20\pi = 2\pi \text{ feet} \approx 6.28 \text{ feet}$$

The length of the arc is approximately 6.28 feet.



## UNIT 3 • CIRCLES AND VOLUME

## Lesson 4: Finding Arc Lengths and Areas of Sectors



## Practice 3.4.1: Defining Radians

Use your knowledge of radian measures to complete the following problems.

1. Convert  $260^\circ$  to radians.
2. Convert 4.9 radians to degrees. Round your answer to the nearest tenth of a degree.
3. A circle has a radius of 2 units. Find the radian measure of a central angle that intercepts an arc length of 5.8 units.
4. A circle has a diameter of 14 units. Find the length of an arc intercepted by a central angle measuring 4 radians.
5. A circle has a radius of 21 units. To the nearest degree, what is the measure of a central angle that intercepts an arc length of 50.4 units?
6. A central angle of  $62^\circ$  intercepts an arc length of 90 units. What is the radius of the circle, rounded to the nearest hundredth?
7. A Ferris wheel has 16 seats and a diameter of 60 feet. The wheel lets off the passengers in one seat, and then revolves until the next seat is at the platform. Approximately how far does each seat travel in this time?
8. How many radians does the hour hand on a clock travel through from 2 to 10?
9. A 20-inch diameter bicycle tire rotates 300 times. How many feet does the bicycle travel?
10. In your own words, what is a radian?

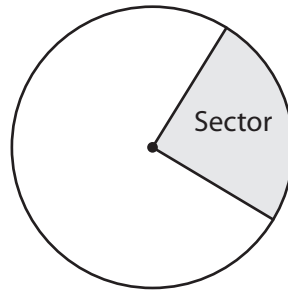
## Lesson 3.4.2: Deriving the Formula for the Area of a Sector

### Introduction

A **sector** is the portion of a circle bounded by two radii and their intercepted arc. Previously, we thought of arc length as a fraction of the circumference of the circle. In a similar way, we can think of a sector as a fraction of the area of the circle. In the same way that we found arc length, we can set up proportions to find the area of a sector.

### Key Concepts

- A sector is the portion of a circle bounded by two radii and their intercepted arc.



- To find the area of a sector,  $A_{\text{sector}}$ , when the central angle  $\theta$  is given in radians, we can set up a proportion using the area of a circle,  $A = \pi r^2$ .

$$\frac{A_{\text{sector}}}{\pi r^2} = \frac{\theta}{2\pi}$$

- We can solve this proportion for the area of the sector and simplify to get a formula for the area of a sector in terms of the radius of the circle and the radian measure of the central angle  $\theta$ .

$$A_{\text{sector}} = \frac{r^2\theta}{2}$$

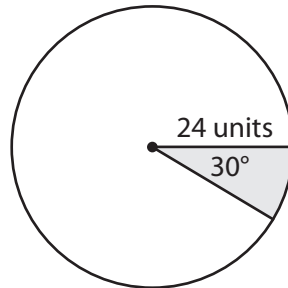
- To find the area of a sector when the central angle is given in degrees, we can set up a proportion using the area of a circle.

$$\frac{A_{\text{sector}}}{A_{\text{circle}}} = \frac{\text{degree measure}}{360^\circ}$$

## Guided Practice 3.4.2

### Example 1

A circle has a radius of 24 units. Find the area of a sector with a central angle of  $30^\circ$ .



1. Find the area of the circle.

$$\text{area} = \pi \cdot \text{radius}^2$$

$$A = \pi r^2 = \pi \cdot 24^2 = 576\pi \text{ square units}$$

2. Set up a proportion.

$$\frac{\text{degree measure of sector}}{360^\circ} = \frac{\text{area of sector}}{\text{area of circle}}$$

$$\frac{30^\circ}{360^\circ} = \frac{\text{area of sector}}{576\pi \text{ square units}}$$

3. Multiply both sides by the area of the circle to find the area of the sector.

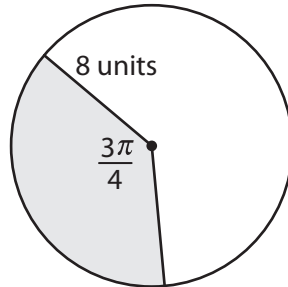
$$\text{area of sector} = \frac{30^\circ}{360^\circ} \cdot 576\pi = 48\pi \text{ square units} \approx 150.80 \text{ square units}$$

The area of the sector is approximately 150.80 units<sup>2</sup>.



## Example 2

A circle has a radius of 8 units. Find the area of a sector with a central angle of  $\frac{3\pi}{4}$  radians.



1. Substitute radius and radian measure into the formula for the area of a sector.

$$\text{area of sector} = \frac{\text{radius}^2 \cdot \text{radian measure of angle}}{2}$$

$$A_{\text{sector}} = \frac{r^2 \theta}{2} = \frac{8^2 \cdot \frac{3\pi}{4}}{2}$$



2. Simplify.

$$A_{\text{sector}} = \frac{8^2 \cdot \frac{3\pi}{4}}{2} = \frac{64 \cdot \frac{3\pi}{4}}{2} = \frac{48\pi}{2} = 24\pi \text{ square units} \approx$$

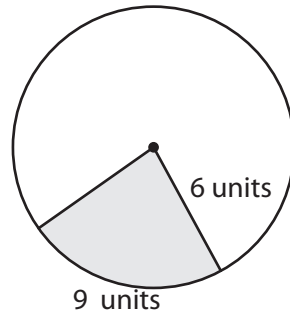
75.40 square units

The area of the sector is approximately 75.40 units<sup>2</sup>.



### Example 3

A circle has a radius of 6 units. Find the area of a sector with an arc length of 9 units.



1. Use the radian measure formula to find the measure of the central angle.

$$\text{radian measure} = \frac{\text{arc length}}{\text{radius}}$$

$$\theta = \frac{s}{r} = \frac{9}{6} = 1.5 \text{ radians}$$



2. Substitute radius and radian measure into the formula for the area of a sector and simplify.

$$A_{\text{sector}} = \frac{r^2\theta}{2} = \frac{6^2 \cdot 1.5}{2} = \frac{54}{2} = 27 \text{ square units}$$

The area of the sector is 27 units<sup>2</sup>.



## UNIT 3 • CIRCLES AND VOLUME

## Lesson 4: Finding Arc Lengths and Areas of Sectors



## Practice 3.4.2: Deriving the Formula for the Area of a Sector

Use your knowledge of the areas of sectors to complete the following problems.

1. Find the area of a sector with a central angle of 7.2 radians and a radius of 14 units.
2. Find the area of a sector with a central angle of  $\frac{2\pi}{3}$  radians and a radius of 6 units.
3. Find the area of a sector with a central angle of  $32^\circ$  and a radius of 8.5 units.
4. A circle has a radius of 10.6 units. Find the area of a sector with an arc length of 15.9 units.
5. A circle has a radius of 4 units. Find the arc length of a sector with an area of 8 square units.
6. A sector has a central angle of  $\frac{\pi}{4}$  radians and an area of 47 square units. What is the area of the circle?
7. A small pizza has a diameter of 10 inches. A slice has a central angle of  $\frac{\pi}{3}$  radians. What is the area of the slice?
8. A pumpkin pie is made in a mini pie pan measuring 5 inches in diameter. It is cut into 4 equal slices. What is the area of 1 piece of pie?
9. A rotating sprinkler sprays a stream of water 32 feet long. The sprinkler rotates  $220^\circ$ . What is the area of the portion of the yard that is watered by the sprinkler?
10. An airplane emits a radar beam that can detect an object up to 70 miles away and covers an angle of  $150^\circ$ . What is the area of the region covered by the radar beam?